

# On Nonhomogeneous Boundary Value Problems for the Stationary Navier–Stokes Equations in 2D Symmetric Semi-Infinite Outlets

M. Chipot<sup>b</sup>, K. Kaulakytė<sup>a,b</sup>, K. Pileckas<sup>a</sup>, and W. Xue<sup>b</sup>

<sup>a</sup>*Faculty of Mathematics and Informatics, Vilnius University, Naugarduko str. 24, LT-03225 Vilnius, Lithuania*

<sup>b</sup>*Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, CH-8057 Zurich, Switzerland*

## Abstract

We study the stationary nonhomogeneous Navier–Stokes problem in a two dimensional symmetric domain with a semi-infinite outlet (for instance, either paraboloidal or channel-like). Under the symmetry assumptions on the domain, boundary value and external force we prove the existence of at least one weak symmetric solution without any restriction on the size of the fluxes, i.e. the fluxes of the boundary value  $\mathbf{a}$  over the inner and the outer boundaries may be arbitrarily large. Only the necessary compatibility condition (the total flux is equal to zero) has to be satisfied. Moreover, the Dirichlet integral of the solution can be finite or infinite depending on the geometry of the domain.

**Keywords:** stationary Navier–Stokes equations; nonhomogeneous boundary value problem; nonzero flux; 2-dimensional noncompact domains, symmetry.

**AMS Subject Classification:** 35Q30; 35J65; 76D03; 76D05.

## 1 Introduction

In this paper we study the steady Navier–Stokes equations with nonhomogeneous boundary conditions

$$\left\{ \begin{array}{lll} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p & = & \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} & = & 0 \quad \text{in } \Omega, \\ \mathbf{u} & = & \mathbf{a} \quad \text{on } \partial\Omega \end{array} \right\} \quad (1.1)$$

in a two dimensional symmetric<sup>1</sup> multiply connected domain  $\Omega$ , having one outlet to infinity (paraboloidal or channel-like), where the vector-valued function  $\mathbf{u} = \mathbf{u}(x)$  is the unknown velocity field, the scalar function  $p = p(x)$  is the pressure of the fluid, while the vector-valued functions  $\mathbf{a} = \mathbf{a}(x)$  and  $\mathbf{f} = \mathbf{f}(x)$  denote the given boundary value and the external force;  $\nu > 0$  is the viscosity constant of the given fluid. The boundary  $\partial\Omega$  consists of an infinite connected outer boundary and finitely many connected components, forming the inner boundary. The fluxes of the boundary value  $\mathbf{a}$  over each component of the inner boundaries and over the outer boundary may be arbitrarily large.

<sup>1</sup>For the definition of a symmetric domain see (2.1).

Let us consider firstly the steady Navier–Stokes problem (1.1) in a bounded domain  $\Omega$  with multiply connected Lipschitz boundary  $\partial\Omega$  consisting of  $N$  disjoint components  $\Gamma_j, j = 1, \dots, N$ . The continuity equation (1.2) implies the necessary compatibility condition for the solvability of the problem (1.1):

$$\int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} dS = \sum_{j=1}^N \int_{\Gamma_j} \mathbf{a} \cdot \mathbf{n} dS = 0, \quad (1.2)$$

where  $\mathbf{n}$  is the unit vector of the outward normal to  $\partial\Omega$ . This condition means that the total flux is zero. Starting from the famous paper of J. Leray published in 1933 (see [20]) the problem (1.1) has been extensively studied. Nevertheless, for a long time the existence of a weak solution  $\mathbf{u} \in W^{1,2}(\Omega)$  to problem (1.1) was only proved either under the condition of zero fluxes

$$\mathbb{F}_j = \int_{\Gamma_j} \mathbf{a} \cdot \mathbf{n} dS = 0, \quad j = 1, 2, \dots, N, \quad (1.3)$$

(e.g., [20], [16], [17], [36]), or assuming the fluxes  $\mathbb{F}_j$  to be sufficiently small (e.g., [2], [3], [4], [8], [15]), or under certain symmetry assumptions on the domain  $\Omega$  and the boundary value  $\mathbf{a}$  (e.g., [1], [5], [6], [25], [29], [30], [31], [12]). We call (1.2) the general outflow condition and (1.3) - the stringent outflow condition. However, the fundamental question (formulated by J. Leray in [20]) whether problem (1.1) is solvable only under the necessary compatibility condition (1.2) (this so called *Leray's problem*) had been open for 80 years. However a huge progress has been made recently. The Leray problem was solved for 2-dimensional bounded multiply connected domain (see [11], [13], [14]).

Nevertheless, not much is known about the nonhomogeneous boundary value problem (1.1) in unbounded domains. The first time in 1999 S.A. Nazarov and K. Pileckas solved problem (1.1) in an infinite layer without the smallness assumption on the flux of the boundary value, i.e. on the bottom of the layer there is a compactly supported sink or source of an arbitrary intensity (see [26]). Later in 2010 J. Neustupa [27], [28] studied problem (1.1) in unbounded domains  $\Omega$  with multiply connected boundaries under the “smallness” assumption of the fluxes of  $\mathbf{a}$  over bounded components of the boundary (he did not impose any conditions on fluxes over infinite parts of  $\partial\Omega$ ). However, the solutions found in [27], [28] have finite Dirichlet integrals (notice that the a priori estimate of solutions was obtained by a contradiction argument). Recently, problem (1.1) has been studied in a class of domains  $\Omega \subset \mathbb{R}^n$ ,  $n = 2, 3$ , having paraboloidal and layer type outlets to infinity (see [9], [10]). In [9], [10] it is assumed that the fluxes of  $\mathbf{a}$  over the bounded connected components of the inner boundary are sufficiently small while there are no restrictions on the fluxes of boundary value  $\mathbf{a}$  over noncompact connected components of the outer boundary. Under these conditions the existence of at least one weak solution to problem (1.1) was proved. This solution can have either finite or infinite Dirichlet integral depending on geometrical properties of the outlets. The proofs in [9], [10] are based on a special construction of the extension of the boundary value  $\mathbf{a}$  which satisfies Leray–Hopf’s inequality and allows to get effective estimate of the solution.

H. Fujita and H. Morimoto (see [21]–[24]) have solved problem (1.1) in a symmetric two dimensional multiply connected domains  $\Omega$  with channel-like outlets to infinity containing a finite number of “holes”. Under certain symmetry assumptions on domain, boundary value and external force, in [21]–[24] the authors also assumed that the boundary value  $\mathbf{a}$  is equal to zero on the outer boundary and that in each outlet the flow

tends to a Poiseuille flow. Moreover, the fluxes over the boundary of each “hole” may be arbitrarily large, but the sum of them has to be equal to the flux of the corresponding Poiseuille flow which needs to be sufficiently small. In addition the viscosity of the fluid has to be relatively large.

In this paper we prove the existence of at least one weak symmetric solution to problem (1.1) in a symmetric domain  $\Omega \subset \mathbb{R}^2$  with either a paraboloidal or a channel-like outlet to infinity assuming that the boundary value  $\mathbf{a}$  and the external force  $\mathbf{f}$  are symmetric functions. Notice that we do not impose any restrictions on the size of the fluxes over both the inner and the outer boundaries.

## 2 Main Notation and Auxiliary Results

Vector valued functions are denoted by bold letters while function spaces for scalar and vector valued functions are denoted the same way.

Let  $\Omega$  be a domain in  $\mathbb{R}^n$ .  $C^\infty(\Omega)$  denotes the set of all infinitely differentiable functions defined on  $\Omega$  and  $C_0^\infty(\Omega)$  is the subset of all functions from  $C^\infty(\Omega)$  with compact support in  $\Omega$ . For given nonnegative integers  $k$  and  $q > 1$ ,  $L^q(\Omega)$  and  $W^{k,q}(\Omega)$  denote the usual Lebesgue and Sobolev spaces;  $W^{k-1/q,q}(\partial\Omega)$  is the trace space on  $\partial\Omega$  of functions from  $W^{k,q}(\Omega)$ ;  $W_0^{k,q}(\Omega)$  is the closure of  $C_0^\infty(\Omega)$  with respect to the norm of  $W^{k,q}(\Omega)$ ; if  $\Omega$  is an unbounded domain, we write  $u \in W_{loc}^{k,q}(\overline{\Omega})$  if  $u \in W^{k,q}(\Omega \cap B_R(0))$  for any  $B_R(0) = \{x \in \mathbb{R}^2 : |x| \leq R\}$ .

Let  $D(\Omega)$  be the Hilbert space of vector valued functions formed as the closure of  $C_0^\infty(\Omega)$  with respect to the Dirichlet norm  $\|\mathbf{u}\|_{D(\Omega)} = \|\nabla \mathbf{u}\|_{L^2(\Omega)}$  induced by the scalar product

$$(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx,$$

where  $\nabla \mathbf{u} : \nabla \mathbf{v} = \sum_{j=1}^n \nabla u_j \cdot \nabla v_j = \sum_{j=1}^n \sum_{k=1}^n \frac{\partial u_j}{\partial x_k} \frac{\partial v_j}{\partial x_k}$ . Denote by  $J_0^\infty(\Omega)$  the set of all solenoidal ( $\operatorname{div} \mathbf{u} = 0$ ) vector fields  $\mathbf{u}$  from  $C_0^\infty(\Omega)$ . By  $H(\Omega)$  we indicate the space formed as the closure of  $J_0^\infty(\Omega)$  with respect to the Dirichlet norm. For any bounded domain  $\Omega$ ,  $H^*(\Omega)$  denotes the dual of  $H(\Omega)$ .  $\|\cdot\|_{H^*(\Omega)}$  denotes the strong dual norm in  $H^*(\Omega)$ .

Assume that  $\Omega$  is symmetric with respect to the  $x_1$ -axis, i.e.,

$$(x_1, x_2) \in \Omega \Leftrightarrow (x_1, -x_2) \in \Omega. \quad (2.1)$$

The vector function  $\mathbf{u} = (u_1, u_2)$  is called symmetric with respect to the  $x_1$ -axis if  $u_1$  is an even function of  $x_2$  and  $u_2$  is an odd function of  $x_2$ , i.e.

$$u_1(x_1, x_2) = u_1(x_1, -x_2), \quad u_2(x_1, x_2) = -u_2(x_1, -x_2). \quad (2.2)$$

For any set of functions  $V(\Omega)$  defined in the symmetric domain  $\Omega$  satisfying (2.1), we denote by  $V_S(\Omega)$  the subspace of symmetric functions from  $V(\Omega)$  satisfying (2.2).

Below we use the well known results which are formulated in the following two lemmas.

**Lemma 2.1.** (see [16]) *Let  $\Pi \subset \mathbb{R}^2$  be a bounded domain with Lipschitz boundary  $\partial\Pi$ . Then for any  $\mathbf{w} \in W^{1,2}(\Pi)$  with  $\mathbf{w}|_{\mathcal{L}} = 0$ ,  $\mathcal{L} \subseteq \partial\Pi$ ,  $\operatorname{meas}(\mathcal{L}) > 0$ , the following inequality*

$$\int_{\Pi} \frac{|\mathbf{w}|^2 \, dx}{\operatorname{dist}^2(x, \mathcal{L})} \leq c \int_{\Pi} |\nabla \mathbf{w}|^2 \, dx \quad (2.3)$$

holds.

**Lemma 2.2.** (see [16]) Let  $\Pi \subset \mathbb{R}^2$  be a bounded domain with Lipschitz boundary  $\partial\Pi$ ,  $\mathcal{L} \subseteq \partial\Pi$ ,  $\text{meas}(\mathcal{L}) > 0$  and  $\mathbf{h} \in W^{1/2,2}(\partial\Pi)$  satisfying the conditions  $\int_{\mathcal{L}} \mathbf{h} \cdot \mathbf{n} dS = 0$ ,  $\text{supp } \mathbf{h} \subseteq \mathcal{L}$ . Then  $\mathbf{h}$  can be extended inside  $\Pi$  in the form

$$\mathbf{A}_*(x, \varepsilon) = \left( -\frac{\partial(\chi(x, \varepsilon) \mathbf{E}(x))}{\partial x_2}, \frac{\partial(\chi(x, \varepsilon) \mathbf{E}(x))}{\partial x_1} \right), \quad (2.4)$$

where  $\mathbf{E} \in W^{2,2}(\Pi)$ ,  $\left( -\frac{\partial \mathbf{E}(x)}{\partial x_2}, \frac{\partial \mathbf{E}(x)}{\partial x_1} \right) \Big|_{\partial\Pi} = \mathbf{h}$  and  $\chi$  is a Hopf's type cut-off function, i.e.  $\chi(x, \varepsilon)$  is smooth,  $\chi(x, \varepsilon) = 1$  on  $\mathcal{L}$ ,  $\text{supp } \chi$  is contained in a small neighborhood of  $\mathcal{L}$  and

$$|\nabla \chi(x, \varepsilon)| \leq \frac{\varepsilon c}{\text{dist}(x, \mathcal{L})}.$$

The constant  $c$  is independent of  $\varepsilon > 0$ .

### 3 Formulation of the Problem

We study problem (1.1) in a symmetric domain  $\Omega \subset \mathbb{R}^2$  having one outlet to infinity. Denote by  $D^{(out)}$  the set

$$D^{(out)} = \{x \in \mathbb{R}^2 : |x_2| \leq g(x_1), \ x_1 > R_* > 0\},$$

where  $g = g(x_1)$  is a positive smooth function such that  $g'$ ,  $g g''$  are bounded on the interval  $[R_*, +\infty)$  and satisfies the Lipschitz condition

$$|g(t_1) - g(t_2)| \leq L |t_1 - t_2|, \quad t_1, t_2 \geq R_*$$

with the Lipschitz constant  $L$ .

We call this set “an outlet to infinity”. Depending on the function  $g$  the outlet  $D^{(out)}$  can be paraboloidal type or channel-like outlet ( $D^{(out)}$  is a channel-like outlet if  $g(x_1) = \text{const}$ ).

Let us take a small positive number  $\gamma$  and introduce another outlet

$$D^{(in)} = \{x \in D^{(out)} : |x_2| \leq \frac{\gamma}{\gamma+1} g(x_1), \ x_1 > R_* > 0\}.$$

We consider an unbounded symmetric domain

$$\Omega = \Omega_0 \cup D,$$

where  $\Omega_0 = \Omega \cap B_{R_0}(0)$  is the bounded part of the domain  $\Omega$  and the unbounded part  $D$  is such that (see Fig. 1)

$$D^{(in)} \subset D \subset D^{(out)}.$$

We assume that

- (i) the bounded domain  $\Omega_0$  has the form

$$\Omega_0 = G_0 \setminus \bigcup_{i=1}^N \overline{G_i},$$

where  $G_0$  and  $G_i$ ,  $i = 1, \dots, N$ , are bounded simply connected domains such that  $\overline{G_i} \subset G_0$  and  $G_N$  denotes the nearest “hole” to the outlet. Denote  $\partial G_i = \Gamma_i$ ,  $i = 1, \dots, N$ ;

(ii) the boundary  $\partial\Omega$  is composed of the bounded connected components  $\Gamma_i$ ,  $i = 1, \dots, N$ , and the infinite component  $\Gamma_0^* = \partial\Omega \setminus \bigcup_{i=1}^N \Gamma_i$ .  $\Gamma_0^*$  can be regarded as the outer boundary of  $\Omega$ , while  $\Gamma_i$ ,  $i = 1, \dots, N$ , as the inner boundaries. Denote  $\Gamma_0 = \Gamma_0^* \cap \partial\Omega_0$ . We suppose that each  $\Gamma_i$ ,  $i = 0, \dots, N$ , intersects the  $x_1$  axis. Below we will use the following notation:

$$R_{l+1} = R_l + \frac{g(R_l)}{2L}, \quad l \geq 0,$$

$$D_l = \{x \in D : x_1 < R_l\}, \quad \Omega_l = \Omega_0 \cup D_l, \quad \omega_l = \Omega_{l+1} \setminus \overline{\Omega}_l.$$

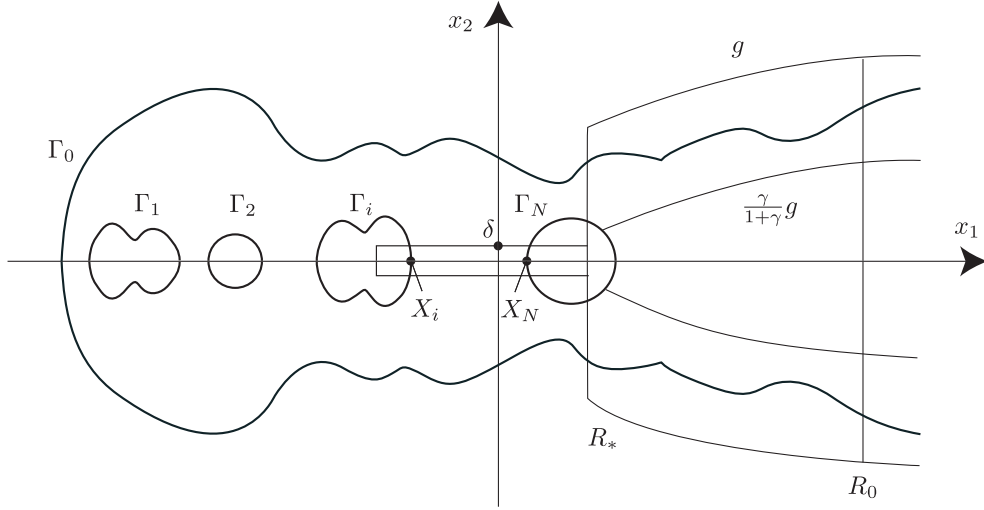


Figure 1: The domain  $\Omega$

**Remark 3.1.** There holds the relation

$$\frac{1}{2}g(R_k) \leq g(t) \leq \frac{3}{2}g(R_k), \quad t \in [R_k, R_{k+1}]. \quad (3.1)$$

Indeed, for  $t \in [R_k, R_{k+1}]$  one has

$$\begin{aligned} -\frac{g(R_k)}{2} &= -L(R_{k+1} - R_k) \leq -L(t - R_k) \leq g(t) - g(R_k) \\ &\leq L(t - R_k) \leq L(R_{k+1} - R_k) = L \frac{g(R_k)}{2L}. \end{aligned}$$

This implies (3.1).

We suppose that the boundary value  $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$  has a compact support. Let

$$\int_{\Gamma_i} \mathbf{a} \cdot \mathbf{n} dS = \mathbb{F}_i, \quad i = 0, \dots, N,$$

be the fluxes of the boundary value  $\mathbf{a}$  over  $\partial\Omega$ . Since the total flux has to be equal to zero (the necessary flux compatibility condition), we have

$$\int_{\sigma(R)} \mathbf{u} \cdot \mathbf{n} dS = - \sum_{i=0}^N \mathbb{F}_i, \quad R > R_0 > 0,$$

where  $\sigma(R)$  is a cross section of the outlet  $D$ .

Two main purposes of this paper are:

1) to construct a suitable symmetric extension  $\mathbf{A}$  of the boundary data  $\mathbf{a}$  which satisfies the Leray–Hopf type inequalities

$$\begin{aligned} \left| \int_{\Omega_{k+1}} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{A} dx \right| &\leq \varepsilon \int_{\Omega_{k+1}} |\nabla \mathbf{w}|^2 dx, \\ \left| \int_{\omega_k} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{A} dx \right| &\leq \varepsilon \int_{\omega_k} |\nabla \mathbf{w}|^2 dx, \end{aligned} \tag{3.2}$$

where  $\mathbf{w} \in W_{loc}^{1,2}(\bar{\Omega})$  is an arbitrary solenoidal function with  $\mathbf{w}|_{\partial\Omega} = 0$  and  $\varepsilon$  can be chosen arbitrary small;

2) to prove the existence of at least one weak symmetric solution  $\mathbf{u}$  to problem (1.1).

## 4 Construction of the Extension

We construct a symmetric extension  $\mathbf{A}$  of the boundary value  $\mathbf{a}$  as a sum:

$$\mathbf{A} = \mathbf{B}_0 + \mathbf{B}_\infty.$$

In order to construct an extension  $\mathbf{B}_0$  we “remove” the fluxes  $\mathbb{F}_i$ ,  $i = 0, \dots, N-1$ , to the boundary  $\Gamma_N$  and then we extend the modified boundary value which has zero fluxes on  $\Gamma_i$ ,  $i = 0, \dots, N-1$ , into  $\Omega$ . After this step we get the flux  $\sum_{i=0}^N \mathbb{F}_i$  on  $\Gamma_N$ . Then by removing it to infinity and extending the modified boundary value from  $\Gamma_N$  into  $\Omega$  we construct the extension  $\mathbf{B}_\infty$ . In general if the stringent outflow condition is not valid one cannot expect that there exists such an extension (see the counterexample in [35]). However, under our symmetry assumptions such an extension can be constructed. The first part of the construction is inspired by some ideas of Fujita [5] and the second part - by techniques proposed in [32].

### 4.1 Construction of the Extension $\mathbf{B}_0$ .

Before we start to construct the extension  $\mathbf{B}_0$  we introduce some auxiliary functions. For  $x \in D^{(out)}$ ,  $x_2 > 0$ , we set (see [32])

$$\xi(x) = \xi(x_1, x_2) = \Psi \left( \varepsilon \ln \frac{\gamma(g(x_1) - x_2)}{x_2} \right), \tag{4.1}$$

where  $0 \leq \Psi \leq 1$  is a smooth monotone cut-off function:

$$\Psi(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t \geq 1. \end{cases}$$

**Lemma 4.1.**  $\xi$  is a smooth function vanishing near  $x_2 = g(x_1)$  and equal to 1 in a neighbourhood of  $x_2 = 0$ . Moreover it holds

$$\left| \frac{\partial \xi}{\partial x_i} \right| \leq \frac{c\varepsilon}{x_2}, \quad i = 1, 2, \quad (4.2)$$

$$\left| \frac{\partial \xi}{\partial x_i} \right| \leq \frac{C(\varepsilon)}{g(x_1)}, \quad \left| \frac{\partial^2 \xi}{\partial x_i \partial x_j} \right| \leq \frac{C(\varepsilon)}{g^2(x_1)} \quad i, j = 1, 2, \quad (4.3)$$

where  $c$  is independent of  $\varepsilon$  and  $C(\varepsilon)$  denotes a constant depending on  $\varepsilon$ .

*Proof.* First one notices that the support of  $\nabla \xi$  is contained in the set where

$$1 \leq \frac{\gamma(g(x_1) - x_2)}{x_2} \leq e^{1/\varepsilon} \Leftrightarrow \frac{(1 + \gamma)x_2}{\gamma} \leq g(x_1) \leq \frac{(e^{1/\varepsilon} + \gamma)x_2}{\gamma}. \quad (4.4)$$

Then since  $\xi(x) = \Psi(\varepsilon \ln(g(x_1) - x_2) - \varepsilon \ln x_2 + \varepsilon \ln \gamma)$  one gets

$$\frac{\partial \xi}{\partial x_1} = \Psi' \cdot \frac{\varepsilon g'(x_1)}{g(x_1) - x_2}, \quad \frac{\partial \xi}{\partial x_2} = \Psi' \cdot \varepsilon \left( \frac{-1}{g(x_1) - x_2} - \frac{1}{x_2} \right), \quad (4.5)$$

where  $\Psi'$  is taken at the point  $\varepsilon \ln \frac{\gamma(g(x_1) - x_2)}{x_2}$ . Since we assumed that  $g'$  is bounded and  $\Psi'$  is bounded as well one derives from (4.4), (4.5)

$$\left| \frac{\partial \xi}{\partial x_1} \right| \leq \frac{c\varepsilon}{x_2}, \quad \left| \frac{\partial \xi}{\partial x_2} \right| \leq c\varepsilon \left( \frac{1}{g(x_1) - x_2} + \frac{1}{x_2} \right) \leq \frac{c\varepsilon}{x_2}$$

for some constant  $c$  and

$$\left| \frac{\partial \xi}{\partial x_i} \right| \leq \frac{c\varepsilon}{x_2} \leq \frac{c\varepsilon(\gamma + e^{1/\varepsilon})}{\gamma g(x_1)}, \quad i = 1, 2.$$

Thus, it remains only to prove the last inequality of (4.3). Differentiating (4.5) we get

$$\frac{\partial^2 \xi}{\partial x_1^2} = \Psi'' \left( \frac{\varepsilon g'(x_1)}{g(x_1) - x_2} \right)^2 + \Psi' \varepsilon \left( \frac{g''(g(x_1) - x_2) - g'^2}{(g(x_1) - x_2)^2} \right).$$

Since we assumed that  $g''g$  is bounded, by (4.4) we obtain

$$\left| \frac{\partial^2 \xi}{\partial x_1^2} \right| \leq \frac{c\varepsilon}{(g(x_1) - x_2)^2} \leq \frac{c\varepsilon}{x_2^2} \leq \frac{c\varepsilon \left( \frac{\gamma + e^{1/\varepsilon}}{\gamma} \right)^2}{g^2(x_1)} = \frac{C(\varepsilon)}{g^2(x_1)}.$$

Similarly we have

$$\begin{aligned} \left| \frac{\partial^2 \xi}{\partial x_1 \partial x_2} \right| &= \left| \Psi'' \varepsilon^2 \frac{g'(x_1)}{g(x_1) - x_2} \cdot \left( -\frac{1}{g(x_1) - x_2} - \frac{1}{x_2} \right) + \Psi' \varepsilon \frac{g'(x_1)}{(g(x_1) - x_2)^2} \right| \\ &\leq \frac{c\varepsilon}{x_2^2} \leq \frac{C(\varepsilon)}{g^2(x_1)}, \\ \left| \frac{\partial^2 \xi}{\partial x_2^2} \right| &= \left| \Psi'' \varepsilon^2 \left( -\frac{1}{g(x_1) - x_2} - \frac{1}{x_2} \right)^2 + \Psi' \varepsilon \left( -\frac{1}{(g(x_1) - x_2)^2} + \frac{1}{x_2^2} \right) \right| \\ &\leq \frac{c\varepsilon}{x_2^2} \leq \frac{C(\varepsilon)}{g^2(x_1)}. \end{aligned}$$

This completes the proof of the Lemma. □

We set

$$\tilde{\boldsymbol{\xi}}(x) = (\tilde{\xi}_1, \tilde{\xi}_2) = \begin{cases} \left( -\frac{\partial \xi(x_1, x_2)}{\partial x_2}, \frac{\partial \xi(x_1, x_2)}{\partial x_1} \right), & x_2 > 0, \\ \left( -\frac{\partial \xi(x_1, -x_2)}{\partial x_2}, -\frac{\partial \xi(x_1, -x_2)}{\partial x_1} \right), & x_2 < 0. \end{cases} \quad (4.6)$$

Then we have

**Lemma 4.2.**  *$\tilde{\boldsymbol{\xi}}$  is a smooth solenoidal symmetric vector field such that for any cross section  $\sigma^{(out)}$  of  $D^{(out)}$  one has*

$$\int_{\sigma^{(out)}} \tilde{\boldsymbol{\xi}} \cdot \mathbf{e}_1 dx_2 = 2. \quad (4.7)$$

*Proof.* Since function  $\frac{\partial \xi}{\partial x_2}$  is even in  $x_2$  we obtain

$$\int_{\sigma^{(out)}} \tilde{\boldsymbol{\xi}} \cdot \mathbf{e}_1 dx_2 = 2 \int_0^{g(x_1)} \left( -\frac{\partial \xi}{\partial x_2} \right) dx_2 = -2\xi(x_1, g(x_1)) + 2\xi(x_1, 0) = 2.$$

□

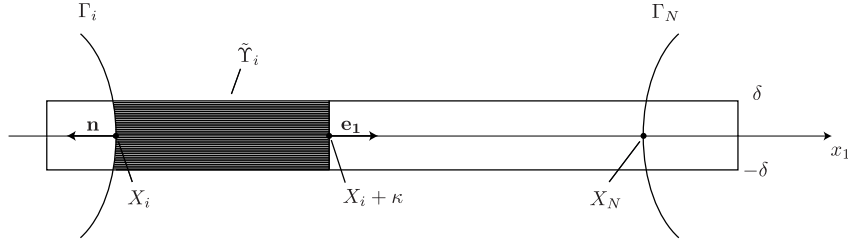


Figure 2: The strip  $\Upsilon_i$

Now we start to construct the extension  $\mathbf{B}_0$ . Let us choose  $\delta$  small enough in such a way that the straight line  $x_2 = \delta$  cuts each of the  $\Gamma_i$ ,  $i = 1, \dots, N$ , at only two points. For  $i = 0, \dots, N-1$  we define the thin strips  $\Upsilon_i = [X_i - \eta_i, X_N + \eta_N] \times [-\delta, \delta]$ , where  $\eta_i$  and  $\eta_N$  are small positive numbers and note that the points  $(X_i - \eta_i, 0)$  and  $(X_N + \eta_N, 0)$  are outside of the domain  $\Omega$  (see Fig. 1). Then on each strip  $\Upsilon_i \cap \Omega$ ,  $i = 0, \dots, N-1$ , joining  $\Gamma_i$  to  $\Gamma_N$  we define  $\mathbf{b}_i$  in the following way

$$\mathbf{b}_i(x) = -\mathbb{F}_i(\tilde{\xi}_1, \tilde{\xi}_2) = \begin{cases} \frac{\mathbb{F}_i}{2} \left( \frac{\partial \xi_\delta(x_2)}{\partial x_2}, 0 \right), & \text{in } \Upsilon_i \cap \Omega, \quad x_2 > 0, \\ \frac{\mathbb{F}_i}{2} \left( \frac{\partial \xi_\delta(-x_2)}{\partial x_2}, 0 \right), & \text{in } \Upsilon_i \cap \Omega, \quad x_2 < 0, \\ (0, 0), & \text{in } \bar{\Omega} \setminus (\Upsilon_i \cap \Omega), \end{cases}$$

where

$$\xi_\delta(x) = \Psi \left( \varepsilon \ln \frac{\delta - x_2}{x_2} \right).$$



Notice that the Lemma 4.1 and Lemma 4.2 are valid if we take  $\gamma = 1$  and  $g(x_1) = \delta$ . Since each vector field  $\mathbf{b}_i$  is solenoidal and vanishes on the upper and lower boundaries of  $\Upsilon_i$ , we have

$$\begin{aligned} 0 &= \int_{\tilde{\Upsilon}_i} \operatorname{div} \mathbf{b}_i dx = \int_{\partial \tilde{\Upsilon}_i} \mathbf{b}_i \cdot \mathbf{n} dS = \int_{\Gamma_i} \mathbf{b}_i \cdot \mathbf{n} dS + \int_{(X_i + \kappa) \times [-\delta, \delta]} \mathbf{b}_i \cdot \mathbf{e}_1 dS \\ &= \int_{\Gamma_i} \mathbf{b}_i \cdot \mathbf{n} dS - \frac{\mathbb{F}_i}{2} \cdot 2 \int_0^\delta \left( -\frac{\partial \xi_\delta}{\partial x_2} \right) dx_2 = \int_{\Gamma_i} \mathbf{b}_i \cdot \mathbf{n} dS - \mathbb{F}_i, \quad \forall i = 0, \dots, N-1, \end{aligned}$$

where  $\tilde{\Upsilon}_i$  is the domain enclosed by  $\Gamma_i$ ,  $(X_i + \kappa) \times [-\delta, \delta]$  ( $\kappa$  is a small positive number) and the lines  $x_2 = \delta$ ,  $x_2 = -\delta$  (see Fig. 2). Therefore, it follows that if  $\mathbf{n}$  denotes the unit outward normal to  $\partial\Omega$  on  $\Gamma_i$  one has

$$\int_{\Gamma_i} \mathbf{b}_i \cdot \mathbf{n} dS = \mathbb{F}_i, \quad \forall i = 0, \dots, N-1.$$

We set

$$\mathbf{b} = \sum_{i=0}^{N-1} \mathbf{b}_i.$$

Clearly  $\mathbf{b}$  is a symmetric solenoidal vector field. Moreover for every  $i = 0, \dots, N-1$  one has (note that the flux of  $\mathbf{b}_i$  vanishes on  $\Gamma_j$  for every  $i \neq j$ )

$$\int_{\Gamma_i} (\mathbf{a} - \mathbf{b}) \cdot \mathbf{n} dS = \int_{\Gamma_i} (\mathbf{a} - \mathbf{b}_i) \cdot \mathbf{n} dS = \mathbb{F}_i - \mathbb{F}_i = 0. \quad (4.8)$$

Because of (4.8) there exists (see Lemma 2.2) an extension  $\mathbf{A}_0$  of  $(\mathbf{a} - \mathbf{b}) \Big|_{\bigcup_{i=0}^{N-1} \Gamma_i}$  such

that  $\operatorname{supp} \mathbf{A}_0$  is contained in a small neighborhood of  $\bigcup_{i=0}^{N-1} \Gamma_i$ ,

$$\operatorname{div} \mathbf{A}_0 = 0, \quad \mathbf{A}_0 \Big|_{\bigcup_{i=0}^{N-1} \Gamma_i} = (\mathbf{a} - \mathbf{b}) \Big|_{\bigcup_{i=0}^{N-1} \Gamma_i},$$

and  $\mathbf{A}_0$  satisfies the Leray–Hopf inequalities<sup>2</sup> for every solenoidal function  $\mathbf{w} \in W_{loc}^{1,2}(\overline{\Omega})$  with  $\mathbf{w}|_{\partial\Omega} = 0$

$$\left| \int_{\Omega_{k+1}} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{A}_0 dx \right| \leq c\varepsilon \int_{\Omega_{k+1}} |\nabla \mathbf{w}|^2 dx. \quad (4.9)$$

Notice that the vector field  $\mathbf{A}_0$  is not necessary symmetric. However, since the boundary value  $(\mathbf{a} - \mathbf{b}) \Big|_{\bigcup_{i=0}^{N-1} \Gamma_i}$  is symmetric,  $\mathbf{A}_0$  can be symmetrized to  $\tilde{\mathbf{A}}_0$  where for  $\mathbf{A} = (A_1, A_2)$

we define  $\tilde{\mathbf{A}} = (\tilde{A}_1, \tilde{A}_2)$  as follows:

$$\begin{aligned} \tilde{A}_1(x) &= \frac{1}{2} \left( A_1(x_1, x_2) + A_1(x_1, -x_2) \right), \quad x \in \Omega, \\ \tilde{A}_2(x) &= \frac{1}{2} \left( A_2(x_1, x_2) - A_2(x_1, -x_2) \right), \quad x \in \Omega. \end{aligned} \quad (4.10)$$

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<sup>2</sup>Notice that the integral over  $\omega_k$  is equal to zero since  $\mathbf{A}_0 = 0$  in  $\omega_k$ .

Define then

$$\mathbf{B}_0 = \tilde{\mathbf{A}}_0 + \mathbf{b}.$$

Hence,  $\mathbf{B}_0$  is a symmetric extension of the boundary value  $\mathbf{a}$  from  $\bigcup_{i=0}^{N-1} \Gamma_i$ . Moreover

$$\int_{\Gamma_N} (\mathbf{a} - \mathbf{b}) \cdot \mathbf{n} dS = \sum_{i=0}^N \mathbb{F}_i = \mathbb{F}.$$

It remains to prove that  $\mathbf{B}_0$  satisfies the Leray-Hopf inequalities. It is enough to prove that each  $\mathbf{b}_i$ ,  $i = 0, \dots, N-1$ , satisfies the Leray-Hopf inequalities.

Let  $\mathbf{w} = (w_1, w_2) \in W_{loc}^{1,2}(\bar{\Omega})$ ,  $\mathbf{w}|_{\partial\Omega} = 0$ , be a symmetric and solenoidal vector field. Then

$$\int_{\Omega_{k+1}} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{b}_i dx = -\mathbb{F}_i \int_{\Upsilon_i \cap \bar{\Omega}} \left( w_1 \frac{\partial w_1}{\partial x_1} + w_2 \frac{\partial w_1}{\partial x_2} \right) \tilde{\xi}_1(x_2) dx$$

Since  $\frac{w_1^2(x)}{2} \tilde{\xi}_1(x_2)$  vanishes on the boundary of  $\Upsilon_i \cap \bar{\Omega}$  one has

$$\int_{\Upsilon_i \cap \bar{\Omega}} w_1(x) \frac{\partial w_1(x)}{\partial x_1} \tilde{\xi}_1(x_2) dx = \frac{1}{2} \int_{\Upsilon_i \cap \bar{\Omega}} \frac{\partial (w_1^2(x) \tilde{\xi}_1(x_2))}{\partial x_1} dx = 0.$$

Therefore, using the definition of  $\tilde{\xi}_1$ , applying the estimate (4.2) and the Hardy<sup>3</sup> type inequality one gets

$$\begin{aligned} \left| \int_{\Omega_{k+1}} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{b}_i dx \right| &\leq |\mathbb{F}_i| \int_{\Upsilon_i \cap \Omega} \left| w_2 \frac{\partial w_1}{\partial x_2} \tilde{\xi}_1 \right| dx \\ &\leq c\varepsilon |\mathbb{F}_i| \int_{\Upsilon_i \cap \Omega} \frac{|w_2|}{|x_2|} \left| \frac{\partial w_1}{\partial x_2} \right| dx \leq c\varepsilon |\mathbb{F}_i| \int_{\Omega_{k+1}} |\nabla \mathbf{w}|^2 dx. \end{aligned}$$

Thus, we have proved the following lemma.

**Lemma 4.3.** *Assume that the boundary value  $\mathbf{a}$  is a symmetric function in  $W^{1/2,2}(\partial\Omega)$  having a compact support. Denote by  $\tilde{\mathbf{a}}$  the restriction of  $\mathbf{a}$  to  $\bigcup_{i=0}^{N-1} \Gamma_i$ . Then for every  $\varepsilon > 0$  there exists a symmetric solenoidal extension  $\mathbf{B}_0$  in  $\Omega$  satisfying  $\mathbf{B}_0|_{\bigcup_{i=0}^{N-1} \Gamma_i} = \tilde{\mathbf{a}}$ ,  $\mathbf{B}_0|_{\partial\Omega \setminus \bigcup_{i=0}^{N-1} \Gamma_i} = 0$  and the Leray-Hopf inequalities<sup>4</sup>, i.e., for every symmetric solenoidal function  $\mathbf{w} \in W_{loc}^{1,2}(\bar{\Omega})$  with  $\mathbf{w}|_{\partial\Omega} = 0$  the following estimates*

$$\left| \int_{\Omega_{k+1}} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{B}_0 dx \right| \leq c\varepsilon \int_{\Omega_{k+1}} |\nabla \mathbf{w}|^2 dx \quad (4.11)$$

hold.

**Remark 4.1.** In the case of a bounded domain the vector field  $\mathbf{B}_0$  is a suitable extension of the boundary value  $\mathbf{a}$ , i.e.  $\mathbf{A} = \mathbf{B}_0$ . The idea of the construction of  $\mathbf{B}_0$  is very similar to that of H. Fujita ([5]).

<sup>3</sup>For the application of the Hardy type inequality we used the fact that  $w_2$  vanishes on  $x_2 = 0$ .

<sup>4</sup>Notice that the integral over  $\omega_k$  is equal to zero since  $\mathbf{B}_0 = 0$  in  $\omega_k$ .

## 4.2 Construction of the Extension $\mathbf{B}_\infty$ .

After moving all the fluxes through  $\Gamma_i$ ,  $i = 0, \dots, N-1$ , to the last inner boundary  $\Gamma_N$  we need to drain the flux from  $\Gamma_N$  to infinity. There we consider a function  $g$  as in Lemma 4.1 and suppose that  $\gamma$  is chosen such that

$$\text{the curve } x_2 = \frac{\gamma}{\gamma+1} g(x_1) \text{ crosses } \Gamma_N. \quad (4.12)$$

Let us introduce the vector field

$$\mathbf{b}_\infty(x) = -\mathbb{F} \tilde{\xi} = \begin{cases} \frac{\mathbb{F}}{2} \left( \frac{\partial \xi(x_1, x_2)}{\partial x_2}, -\frac{\partial \xi(x_1, x_2)}{\partial x_1} \right), & x_2 > 0, \\ \frac{\mathbb{F}}{2} \left( \frac{\partial \xi(x_1, -x_2)}{\partial x_2}, \frac{\partial \xi(x_1, -x_2)}{\partial x_1} \right), & x_2 < 0, \end{cases}$$

where  $\xi$  is defined by (4.1) for  $x \in D^{(in)}$  and extended by 0 into  $D$ . Then since for any cross section  $\sigma$

$$\begin{aligned} \int_{\Gamma_N} \mathbf{b}_\infty \cdot \mathbf{n} dS &= - \int_{\sigma} \mathbf{b}_\infty \cdot \mathbf{n} dS = -\frac{\mathbb{F}}{2} \cdot 2 \int_0^{\frac{\gamma}{\gamma+1} g(x_1)} \frac{\partial \xi}{\partial x_2} dx_2 = \\ &= -\mathbb{F} \left( \xi(x_1, \frac{\gamma}{\gamma+1} g(x_1)) - \xi(x_1, 0) \right) = \mathbb{F}, \end{aligned}$$

one has

$$\int_{\Gamma_N} (\mathbf{a} - \mathbf{b} - \mathbf{b}_\infty) \cdot \mathbf{n} dS = 0. \quad (4.13)$$

Because of (4.13) there exists (see Lemma 2.2) an extension  $\mathbf{A}_\infty$  of  $(\mathbf{a} - \mathbf{b} - \mathbf{b}_\infty)|_{\Gamma_N}$  such that  $\text{supp } \mathbf{A}_\infty$  is contained in a small neighborhood of  $\Gamma_N$ ,

$$\text{div } \mathbf{A}_\infty = 0, \quad \mathbf{A}_\infty|_{\Gamma_N} = (\mathbf{a} - \mathbf{b} - \mathbf{b}_\infty)|_{\Gamma_N},$$

and  $\mathbf{A}_\infty$  satisfies the Leray–Hopf inequalities for every solenoidal function  $\mathbf{w} \in W_{loc}^{1,2}(\overline{\Omega})$  with  $\mathbf{w}|_{\partial\Omega} = 0$

$$\begin{aligned} \left| \int_{\Omega_{k+1}} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{A}_\infty dx \right| &\leq c\varepsilon \int_{\Omega_{k+1}} |\nabla \mathbf{w}|^2 dx, \\ \left| \int_{\omega_k} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{A}_\infty dx \right| &\leq c\varepsilon \int_{\omega_k} |\nabla \mathbf{w}|^2 dx \end{aligned} \quad (4.14)$$

with a constant  $c$  independent of  $k$  and  $\varepsilon$ . Notice that the vector field  $\mathbf{A}_\infty$  is not necessary symmetric. However, since the boundary value  $(\mathbf{a} - \mathbf{b} - \mathbf{b}_\infty)|_{\Gamma_N}$  is symmetric,  $\mathbf{A}_\infty$  can be symmetrized to  $\tilde{\mathbf{A}}_\infty$  as in (4.10). Then

$$\mathbf{B}_\infty = \mathbf{b} + \mathbf{b}_\infty + \tilde{\mathbf{A}}_\infty$$

is a symmetric solenoidal extension of  $\mathbf{a}$  on  $\Gamma_N$ . It remains to prove that  $\mathbf{B}_\infty$  satisfies the Leray–Hopf inequalities. It is enough to prove that  $\mathbf{b}_\infty$  satisfies the Leray–Hopf inequalities.

Let  $\mathbf{w} = (w_1, w_2) \in W_{loc}^{1,2}(\bar{\Omega})$ ,  $\mathbf{w}|_{\partial\Omega} = 0$ , be a symmetric and solenoidal vector field. We use the well known identity

$$(\mathbf{w} \cdot \nabla) \mathbf{w} = \nabla\left(\frac{1}{2}|\mathbf{w}|^2\right) + \left(\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}\right) (-w_2, w_1). \quad (4.15)$$

Since  $\mathbf{b}_\infty$  is solenoidal, it is  $L^2$ -orthogonal to the first term of the right-hand side of (4.15). Then one obtains

$$\begin{aligned} \left| \int_{\Omega_{k+1}} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{b}_\infty dx \right| &\leq |\mathbb{F}| \int_{\Omega_{k+1}} \left| \left( \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2} \right) (-w_2 \tilde{\xi}_1 + w_1 \tilde{\xi}_2) \right| dx \\ &\leq |\mathbb{F}| \left( \int_{\Omega_{k+1}} |\nabla \mathbf{w}|^2 dx \right)^{1/2} \left( \int_{\Omega_{k+1}} |(-w_2 \tilde{\xi}_1 + w_1 \tilde{\xi}_2)|^2 dx \right)^{1/2}. \end{aligned} \quad (4.16)$$

Let  $G^\pm$  denotes the curve  $x_2 = \pm g(x_1)$ . Then using (4.2), (4.5) for  $x \in \Omega$  and  $x_2 > 0$ , we have

$$|\tilde{\xi}_1| = \left| \frac{\partial \xi}{\partial x_2} \right| \leq c\varepsilon \frac{1}{x_2}, \quad |\tilde{\xi}_2| = \left| \frac{\partial \xi}{\partial x_1} \right| \leq \frac{c\varepsilon}{\text{dist}(x, G^+)}. \quad (4.17)$$

Therefore, from (4.16) and (4.17) applying<sup>5</sup> Hardy type inequality (see Lemma 2.1) we get

$$\begin{aligned} &\left| \int_{\Omega_{k+1}^+} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{b}_\infty dx \right| \\ &\leq c\varepsilon |\mathbb{F}| \left( \int_{\Omega_{k+1}^+} |\nabla \mathbf{w}|^2 dx \right)^{1/2} \left[ \left( \int_{\Omega_{k+1}^+} \frac{|w_2|^2}{|x_2|^2} dx \right)^{1/2} + \left( \int_{\Omega_{k+1}^+} \frac{|w_1|^2}{\text{dist}^2(x, G^+)} dx \right)^{1/2} \right] \\ &\leq c\varepsilon |\mathbb{F}| \int_{\Omega_{k+1}^+} |\nabla \mathbf{w}|^2 dx, \end{aligned}$$

where  $\Omega_{k+1}^+ = \{x \in \Omega_{k+1} : x_2 > 0\}$ . The same estimate is valid in  $\Omega_{k+1}^-$ . Therefore,  $\mathbf{b}_\infty$  satisfies the Leray-Hopf inequalities for every symmetric solenoidal function  $\mathbf{w} \in W_{loc}^{1,2}(\bar{\Omega})$  with  $\mathbf{w}|_{\partial\Omega} = 0$

$$\begin{aligned} \left| \int_{\Omega_{k+1}} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{b}_\infty dx \right| &\leq c\varepsilon \int_{\Omega_{k+1}} |\nabla \mathbf{w}|^2 dx, \\ \left| \int_{\omega_k} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{b}_\infty dx \right| &\leq c\varepsilon \int_{\omega_k} |\nabla \mathbf{w}|^2 dx. \end{aligned} \quad (4.18)$$

Moreover, one has the estimates

$$|\mathbf{b}_\infty| \leq \frac{C(\varepsilon)}{g(x_1)}, \quad |\nabla \mathbf{b}_\infty| \leq \frac{C(\varepsilon)}{g^2(x_1)}, \quad x \in D. \quad (4.19)$$

Hence together with Lemma 4.3 we proved the following result.

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<sup>5</sup>Here we used the fact that  $w_2 = 0$  on  $x_2 = 0$  and we supposed that  $\mathbf{w}$  is extended by 0 outside  $\Omega$ .

**Lemma 4.4.** Assume that the boundary value  $\mathbf{a}$  is a symmetric function in  $W^{1/2,2}(\partial\Omega)$  having a compact support. Then for every  $\varepsilon > 0$  there exists a symmetric solenoidal extension  $\mathbf{A} = \mathbf{B}_0 + \mathbf{B}_\infty$  in  $\Omega$  satisfying the Leray-Hopf inequalities, i.e., for every symmetric solenoidal function  $\mathbf{w} \in W_{loc}^{1,2}(\overline{\Omega})$  with  $\mathbf{w}|_{\partial\Omega} = 0$  the following estimates

$$\begin{aligned} \left| \int_{\Omega_{k+1}} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{A} \, dx \right| &\leq c\varepsilon \int_{\Omega_{k+1}} |\nabla \mathbf{w}|^2 \, dx, \\ \left| \int_{\omega_k} (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \mathbf{A} \, dx \right| &\leq c\varepsilon \int_{\omega_k} |\nabla \mathbf{w}|^2 \, dx \end{aligned} \quad (4.20)$$

hold. The constant  $c$  is independent of  $k$  and  $\varepsilon$ .

**Remark 4.2.** The constant  $c$  in (4.20) is of the type

$$c_1 \sum_{i=0}^N |\mathbb{F}_i| = c_1 \sum_{i=0}^N \left| \int_{\Gamma_i} \mathbf{a} \cdot \mathbf{n} \, dS \right| \leq c_2 \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)},$$

where  $c_2$  is independent of  $\mathbf{a}$ .

## 5 Existence Theorem

We look for the solution  $\mathbf{u}$  in the form

$$\mathbf{u}(x) = \mathbf{A}(x, \varepsilon) + \mathbf{v}(x), \quad (5.1)$$

where  $\mathbf{A}$  is the symmetric extension of the boundary value  $\mathbf{a}$  constructed in the previous section (see Lemma 4.4).

**Definition 5.1.** Under a weak solution of problem (1.1) we understand a solenoidal vector field  $\mathbf{u}$  which is of the type (5.1) with the symmetric vector field  $\mathbf{v} \in W_{loc}^{1,2}(\overline{\Omega})$  satisfying the following conditions

$$\operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \quad \mathbf{v} = 0 \text{ on } \partial\Omega,$$

and the integral identity

$$\begin{aligned} &\nu \int_{\Omega} \nabla \mathbf{v} : \nabla \boldsymbol{\eta} \, dx - \int_{\Omega} ((\mathbf{A} + \mathbf{v}) \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{v} \, dx - \int_{\Omega} (\mathbf{v} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{A} \, dx \\ &= \int_{\Omega} (\mathbf{A} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{A} \, dx - \nu \int_{\Omega} \nabla \mathbf{A} : \nabla \boldsymbol{\eta} \, dx + \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\eta} \, dx \quad \forall \boldsymbol{\eta} \in J_0^\infty(\Omega). \end{aligned} \quad (5.2)$$

Then we have

**Theorem 5.1.** Suppose that  $\Omega \subset \mathbb{R}^2$  is an unbounded domain symmetric with respect to the  $x_1$  axis and each  $\Gamma_i$ ,  $i = 0, \dots, N$ , intersects the  $x_1$  axis. Assume that the boundary value  $\mathbf{a}$  is a symmetric field in  $W^{1/2,2}(\partial\Omega)$  having a compact support. Let  $\mathbf{f}$  be a distribution which is symmetric in the sense that

$$\langle \mathbf{f}, \boldsymbol{\eta} \rangle = \langle \mathbf{f}, \tilde{\boldsymbol{\eta}} \rangle \quad \forall \boldsymbol{\eta} \in J_0^\infty(\Omega) \quad (5.3)$$

( $\tilde{\boldsymbol{\eta}}$  denotes the symmetrization of  $\boldsymbol{\eta}$  as defined in (4.10) for  $\mathbf{A}$ ) and such that

$$\mathbf{f} \in H^*(\Omega_k) \quad \forall k \quad \text{and} \quad \|\mathbf{f}\|_* = \sup_{k \geq 1} \left( \left( 1 + \int_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)} \right)^{-1/2} \|\mathbf{f}\|_{H^*(\Omega_k)} \right) < +\infty.$$

If either

$$(i) \int_{R_0}^{+\infty} \frac{dx_1}{g^3(x_1)} < +\infty$$

or

$$(ii) \int_{R_0}^{+\infty} \frac{dx_1}{g^3(x_1)} = +\infty \quad \text{and} \quad D = D^{(out)},$$

then the problem (1.1) admits a weak solution  $\mathbf{u} = \mathbf{A} + \mathbf{v}$  in the sense of the definition 5.1. In the case (i) the weak solution  $\mathbf{u}$  satisfies the estimate

$$\int_{\Omega} |\nabla \mathbf{u}|^2 dx \leq c(\mathbf{a}, \|\mathbf{f}\|_*) \left( 1 + \int_{R_0}^{+\infty} \frac{dx_1}{g^3(x_1)} \right) \quad (5.4)$$

and in the case (ii)

$$\int_{\Omega_k} |\nabla \mathbf{u}|^2 dx \leq c(\mathbf{a}, \|\mathbf{f}\|_*) \left( 1 + \int_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)} \right), \quad (5.5)$$

where  $c(\mathbf{a}, \|\mathbf{f}\|_*) = c \left( \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}^2 + \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}^4 + \|\mathbf{f}\|_*^2 \right)$  and  $c$  is independent of  $k$ .

**Remark 5.1.** In the equality (5.2) and in what follows we kept for simplicity the notation of  $\langle \mathbf{f}, \boldsymbol{\eta} \rangle$  as an integral. One should also notice that due to the symmetry assumptions on  $\mathbf{A}$ ,  $\mathbf{v}$ ,  $\mathbf{f}$  the equality (5.2) will hold as soon as it holds for any  $\boldsymbol{\eta} \in J_{0,S}^\infty(\Omega)$ , i.e. for  $\boldsymbol{\eta} \in J_0^\infty(\Omega)$  which is symmetric.

In order to prove the existence of at least one weak solution we need some classical results.

**Lemma 5.1.** (Leray-Schauder theorem). Let  $V$  be a Hilbert space and  $\mathcal{A} : V \rightarrow V$  be a nonlinear compact operator. If the norms of all possible solutions of the operator equation

$$u^{(\lambda)} = \lambda \mathcal{A} u^{(\lambda)}, \quad \lambda \in [0, 1],$$

are bounded with the same constant  $c$  independent of  $\lambda$ , i.e.,

$$\|u^{(\lambda)}\|_V \leq c \quad \forall \lambda \in [0, 1],$$

then the operator equation

$$u = \mathcal{A} u$$

has at least one solution  $u \in V$  (see, for example, [16]).

**Lemma 5.2.** (Poincaré inequality). Let  $u \in W_{loc}^{1,2}(\overline{\Omega})$ ,  $u|_{\partial\Omega} = 0$ . Then the following inequality

$$\int_{\omega_k} |u(x)|^2 dx \leq c g^2(R_k) \int_{\omega_k} |\nabla u(x)|^2 dx, \quad (5.6)$$

holds, where the constant  $c$  is independent of  $u$  and  $k$ .

For the proof of this lemma recall (3.1):

$$\frac{1}{2}g(R_k) \leq g(t) \leq \frac{3}{2}g(R_k), \quad t \in [R_k, R_{k+1}].$$

**Lemma 5.3.** Let  $u \in W_{loc}^{1,2}(\overline{\Omega})$ ,  $u|_{\partial\Omega} = 0$ . Then the following inequality

$$\|u\|_{L^4(\omega_k)} \leq c g^{1/2}(R_k) \|\nabla u\|_{L^2(\omega_k)}, \quad (5.7)$$

holds, where the constant  $c$  is independent of  $u$  and  $k$ .

*Proof.* The proof of this lemma follows directly from the following inequality

$$\|u\|_{L^4(\omega_k)} \leq c \|\nabla u\|_{L^2(\omega_k)}^{1/2} \cdot \|u\|_{L^2(\omega_k)}^{1/2}, \quad (5.8)$$

the estimates (3.1) and the Poincaré inequality (5.6). The constant  $c$  in (5.8) is independent of  $k$ .  $\square$

**Lemma 5.4.** Suppose that  $D = D^{(out)}$ , i.e.  $\omega_k = \{x : R_k < x_1 < R_{k+1}, |x_2| < g(x_1)\}$ . Let  $f \in L^2(\omega_k)$  and

$$\int_{\omega_k} f \, dx = 0.$$

Then the problem

$$\left\{ \begin{array}{l} \operatorname{div} \mathbf{u} = f \quad \text{in } \omega_k, \\ \mathbf{u} = 0 \quad \text{on } \partial\omega_k \end{array} \right\} \quad (5.9)$$

admits a solution  $\mathbf{u} \in W_0^{1,2}(\omega_k)$  satisfying the estimate

$$\|\nabla \mathbf{u}\|_{L^2(\omega_k)} \leq c \|f\|_{L^2(\omega_k)} \quad (5.10)$$

with the constant  $c$  independent of  $\mathbf{u}$ ,  $f$  and  $k$ .

**Remark 5.2.** In [32] the family of the domains  $\omega_k$  was chosen in a special way in order to have solutions of the problem (5.9) satisfying the estimates (5.10) with a constant  $c$  independent of  $k$ . Below we give a detailed proof of that fact.

*Proof.* Recall that  $R_{k+1} - R_k = \frac{g(R_k)}{2L}$  and  $L$  is the Lipschitz constant of  $g$ . Consider the transformation  $F$  defined by

$$y = (y_1, y_2) = F(x) = \left( \frac{2L(x_1 - R_k)}{g(R_k)}, \frac{2Lx_2}{g(R_k)} \right).$$

Through this transformation  $\omega_k$  is transformed into a domain  $F(\omega_k)$  such that

$$\begin{aligned} 0 \leq y_1 &= \frac{2L(x_1 - R_k)}{g(R_k)} \leq \frac{2L(R_{k+1} - R_k)}{g(R_k)} = 1, \\ |y_2| &\leq \frac{2Lg(x_1)}{g(R_k)} = \frac{2L(g(x_1) - g(R_k) + g(R_k))}{g(R_k)} \leq 2L \left( \frac{L(R_{k+1} - R_k)}{g(R_k)} + 1 \right) = 3L. \end{aligned}$$

Moreover, the upper and the lower boundary of  $F(\omega_k)$  is given by  $\pm$  the graph of the function  $h_k$  defined as

$$h_k(y_1) = \frac{2L}{g(R_k)} g\left(\frac{g(R_k)}{2L} y_1 + R_k\right), \quad y_1 \in (0, 1).$$

Note that  $h_k$  satisfies

$$|h_k(y_1) - h_k(y'_1)| \leq L|y_1 - y'_1| \quad \forall y_1, y'_1 \in (0, 1).$$

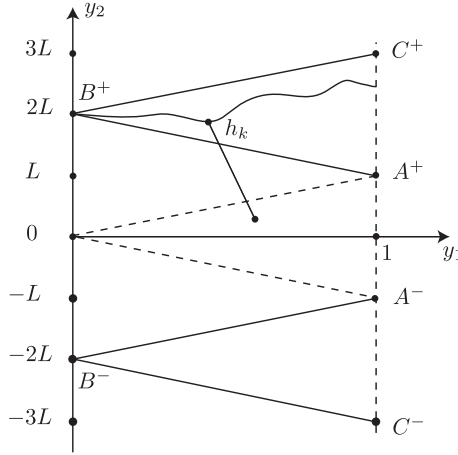


Figure 3: Transformed domain

Since  $h_k(0) = 2L$  it is clear that the graph of  $h_k$  (resp.  $-h_k$ ) is contained in the triangle  $A^+B^+C^+$  (resp.  $A^-B^-C^-$ ) (see Fig. 3). Any straight line joining a point of the triangle  $A^-OA^+$  (notice that  $O = (0,0)$ ) to the graph of  $\pm h_k$  will necessarily have a slope larger than  $L$  and thus  $F(\omega_k)$  is a star shaped domain with respect to any point of  $A^-OA^+$  and bounded independently of  $k$ . One has if  $J_F(x)$  denotes the Jacobian determinant of  $F$  and  $F^{-1}$  the inverse of  $F$

$$\int_{F(\omega_k)} f(F^{-1}(y)) dy = \int_{\omega_k} f(x) |J_F(x)| dx = \left( \frac{2L}{g(R_k)} \right)^2 \int_{\omega_k} f(x) dx = 0.$$

Thus there exists  $\mathbf{v}$  solution to

$$\left\{ \begin{array}{l} \operatorname{div} \mathbf{v}(y) = \frac{g(R_k)}{2L} f(F^{-1}(y)) \text{ in } F(\omega_k), \\ \mathbf{v}(y) = 0 \text{ on } \partial F(\omega_k) \end{array} \right\}$$

which satisfies (see [18])

$$\|\nabla \mathbf{v}\|_{L^2(F(\omega_k))} \leq c \left\| \frac{g(R_k)}{2L} f(F^{-1}(y)) \right\|_{L^2(F(\omega_k))}, \quad (5.11)$$

where  $c$  is independent of  $k$ . Set

$$\mathbf{u}(x) = \mathbf{v}(F(x)).$$

One has the summation convention

$$\frac{\partial u_k(x)}{\partial x_i} = \sum_{l=1}^2 \frac{\partial v_k(F(x))}{\partial y_l} \cdot \frac{\partial y_l}{\partial x_i} = \frac{2L}{g(R_k)} \cdot \frac{\partial v_k(F(x))}{\partial y_i}.$$

Thus  $\mathbf{u}$  satisfies

$$\left\{ \begin{array}{l} \operatorname{div} \mathbf{u}(x) = \frac{2L}{g(R_k)} \operatorname{div} \mathbf{v}(F(x)) = f(x) \text{ in } \omega_k, \\ \mathbf{u}(x) = 0 \text{ on } \partial \omega_k. \end{array} \right\}$$

Moreover,

$$\|\nabla \mathbf{u}\|_{L^2(\omega_k)} = \frac{2L}{g(R_k)} \|\nabla \mathbf{v}(F(x))\|_{L^2(\omega_k)}.$$



Since (see (5.11))

$$\begin{aligned} \|\nabla \mathbf{v}(F(x))\|_{L^2(\omega_k)}^2 &= \int_{\omega_k} |\nabla \mathbf{v}(F(x))|^2 dx = \int_{F(\omega_k)} |\nabla \mathbf{v}(y)|^2 \left(\frac{g(R_k)}{2L}\right)^2 dy \\ &\leq c \left(\frac{g(R_k)}{2L}\right)^4 \int_{F(\omega_k)} f^2(F^{-1}(y)) dy = c \left(\frac{g(R_k)}{2L}\right)^2 \int_{\omega_k} f^2(x) dx \end{aligned}$$

the result follows.  $\square$

*Proof of the Theorem 5.1:* we construct a solution to (5.2) as limit of a sequence  $\mathbf{v}^{(l)} \in H_S(\Omega_l)$ , where  $\mathbf{v}^{(l)}$  are solutions to

$$\begin{aligned} &\nu \int_{\Omega_l} \nabla \mathbf{v}^{(l)} : \nabla \boldsymbol{\eta} dx - \int_{\Omega_l} ((\mathbf{A} + \mathbf{v}^{(l)}) \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{v}^{(l)} dx - \int_{\Omega_l} (\mathbf{v}^{(l)} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{A} dx \\ &= \int_{\Omega_l} (\mathbf{A} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{A} dx - \nu \int_{\Omega_l} \nabla \mathbf{A} : \nabla \boldsymbol{\eta} dx + \int_{\Omega_l} \mathbf{f} \cdot \boldsymbol{\eta} dx \end{aligned} \quad (5.12)$$

for any test function  $\boldsymbol{\eta} \in H_S(\Omega_l)$ . Due, for instance, to the Riesz representation theorem there exists a unique element  $\mathcal{A} \mathbf{v}^{(l)} \in H_S(\Omega_l)$  such that

$$\begin{aligned} \int_{\Omega_l} \nabla \widehat{\mathcal{A}} \mathbf{v}^{(l)} : \nabla \boldsymbol{\eta} dx &= \nu^{-1} \left( \int_{\Omega_l} (\mathbf{v}^{(l)} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{v}^{(l)} dx + \int_{\Omega_l} (\mathbf{A} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{v}^{(l)} dx \right. \\ &\quad \left. + \int_{\Omega_l} (\mathbf{v}^{(l)} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{A} dx + \int_{\Omega_l} (\mathbf{A} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{A} dx + \int_{\Omega_l} \mathbf{f} \cdot \boldsymbol{\eta} dx \right) - \int_{\Omega_l} \nabla \mathbf{A} : \nabla \boldsymbol{\eta} dx \quad \forall \boldsymbol{\eta} \in H_S(\Omega_l). \end{aligned}$$

The equation (5.12) is equivalent to the operator equation

$$\mathbf{v}^{(l)} = \widehat{\mathcal{A}} \mathbf{v}^{(l)}. \quad (5.13)$$

It can be proved (see [16]) that the operator  $\widehat{\mathcal{A}} : H_S(\Omega_l) \hookrightarrow H_S(\Omega_l)$  is compact and the solvability of the operator equations (5.13) can be obtained by applying the Leray–Schauder Theorem. To do this we need to show that the norms of all possible solutions of the operator equations

$$\mathbf{v}^{(l, \lambda)} = \lambda \widehat{\mathcal{A}} \mathbf{v}^{(l, \lambda)}, \quad \lambda \in [0, 1], \quad (5.14)$$

are bounded by a constant independent of  $\lambda$ . Take  $\boldsymbol{\eta} = \mathbf{v}^{(l, \lambda)}$  in (5.14). This yields

$$\begin{aligned} \nu \int_{\Omega_l} |\nabla \mathbf{v}^{(l, \lambda)}|^2 dx &= \lambda \int_{\Omega_l} (\mathbf{A} \cdot \nabla) \mathbf{v}^{(l, \lambda)} \cdot \mathbf{A} dx - \lambda \nu \int_{\Omega_l} \nabla \mathbf{A} : \nabla \mathbf{v}^{(l, \lambda)} dx \\ &\quad + \lambda \int_{\Omega_l} \mathbf{f} \cdot \mathbf{v}^{(l, \lambda)} dx + \lambda \int_{\Omega_l} (\mathbf{v}^{(l, \lambda)} \cdot \nabla) \mathbf{v}^{(l, \lambda)} \cdot \mathbf{A} dx. \end{aligned} \quad (5.15)$$

We estimate the first three terms of the right-hand side of (5.15) by using the Hölder and the Cauchy inequalities, and to estimate the last term of (5.15) we use the Leray–Hopf inequality (4.20). We obtain

$$\begin{aligned} \nu \int_{\Omega_l} |\nabla \mathbf{v}^{(l, \lambda)}|^2 dx &\leq c \mu \int_{\Omega_l} |\nabla \mathbf{v}^{(l, \lambda)}|^2 dx \\ &\quad + \frac{c}{\mu} \left( \int_{\Omega_l} |\nabla \mathbf{A}|^2 dx + \int_{\Omega_l} |\mathbf{A}|^4 dx + \|\mathbf{f}\|_{H^*(\Omega_l)}^2 \right) + c(\mathbb{F}_1, \dots, \mathbb{F}_N) \varepsilon \int_{\Omega_l} |\nabla \mathbf{v}^{(l, \lambda)}|^2 dx. \end{aligned} \quad (5.16)$$

Taking  $\mu = \frac{\nu}{4c}$  and  $\varepsilon = \frac{\nu}{4c(\mathbb{F}_1, \dots, \mathbb{F}_N)}$ , we obtain

$$\frac{\nu}{2} \|\nabla \mathbf{v}^{(l, \lambda)}\|_{L^2(\Omega_l)}^2 \leq c \left( \|\nabla \mathbf{A}\|_{L^2(\Omega_l)}^2 + \|\mathbf{A}\|_{L^4(\Omega_l)}^4 + \|\mathbf{f}\|_{H^*(\Omega_l)}^2 \right).$$

Since  $\varepsilon$  is now fixed, we have also (note that  $\text{supp } \mathbf{B}_0 \subset \Omega_0$ )

$$\begin{aligned} \|\nabla \mathbf{A}\|_{L^2(\Omega_l)}^2 &= \|\nabla \mathbf{B}_0 + \nabla \mathbf{B}_\infty\|_{L^2(\Omega_l)}^2 = \|\nabla \mathbf{B}_0\|_{L^2(\Omega_0)}^2 + \|\nabla \mathbf{B}_\infty\|_{L^2(\Omega_l)}^2 \\ &\leq c \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}^2 \left( \int_{\Omega_0} dx + \int_{\Omega_l} \frac{dx}{g^4(x_1)} \right) \\ &\leq c \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}^2 \left( \text{meas}(\Omega_0) + \int_{R_0}^{R_l} \int_{-g(x_1)}^{g(x_1)} \frac{dx_1 dx_2}{g^4(x_1)} \right) \\ &\leq c \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}^2 \left( 1 + \int_{R_0}^{R_l} \frac{dx_1}{g^3(x_1)} \right). \end{aligned} \tag{5.17}$$

Similarly (see (4.19))

$$\|\mathbf{A}\|_{L^4(\Omega_l)}^4 \leq c \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}^4 \left( 1 + \int_{R_0}^{R_l} \frac{dx_1}{g^3(x_1)} \right). \tag{5.18}$$

The constant  $c$  in (5.17) and (5.18) is independent of  $l$ .

Therefore, we obtain for all  $0 \leq \lambda \leq 1$

$$\|\nabla \mathbf{v}^{(l, \lambda)}\|_{L^2(\Omega_l)}^2 \leq c(\mathbf{a}, \|\mathbf{f}\|_*) \left( 1 + \int_{R_0}^{R_l} \frac{dx_1}{g^3(x_1)} \right).$$

Hence, according to the Leray–Schauder Theorem each operator equation (5.13) has at least one weak symmetric solution  $\mathbf{v}^{(l)} \in H_S(\Omega_l)$ . These solutions satisfy the integral identity (5.12) and the inequality

$$\|\nabla \mathbf{v}^{(l)}\|_{L^2(\Omega_l)}^2 \leq c(\mathbf{a}, \|\mathbf{f}\|_*) \left( 1 + \int_{R_0}^{R_l} \frac{dx_1}{g^3(x_1)} \right). \tag{5.19}$$

If  $\int_{R_0}^{+\infty} \frac{dx_1}{g^3(x_1)} < +\infty$ , the right hand side of the above inequality is bounded by a constant uniformly independent of  $l$ . Extending the solutions  $\mathbf{v}^{(l)}$  by 0 into  $\Omega \setminus \Omega_l$  we get functions  $\tilde{\mathbf{v}}^{(l)} \in H_S(\Omega)$ . The sequence  $\{\tilde{\mathbf{v}}^{(l)}\}$  is bounded in the space  $H_S(\Omega)$ . Therefore, there exists a subsequence  $\{\tilde{\mathbf{v}}^{(l_m)}\}$  which converges weakly in  $H_S(\Omega)$  and strongly<sup>6</sup> in  $L^4(\Omega_l)$  for any  $l$ . Taking in integral identity (5.12) an arbitrary test function  $\boldsymbol{\eta}$  with a compact support, we can find a number  $l$  such that  $\text{supp } \boldsymbol{\eta} \subset \Omega_l$  and  $\boldsymbol{\eta} \in H_S(\Omega_l)$ . We can pass in (5.12) to a limit as  $l_m \rightarrow +\infty$ . As a result we get for the limit vector function  $\mathbf{v}$  the integral identity (5.2). Obviously, following estimate

$$\int_{\Omega} |\nabla \mathbf{v}|^2 dx \leq c(\mathbf{a}, \|\mathbf{f}\|_*) \left( 1 + \int_{R_0}^{+\infty} \frac{dx_1}{g^3(x_1)} \right)$$

---

<sup>6</sup>Notice that the embedding  $H_S(\Omega_l) \hookrightarrow L^4_S(\Omega_l)$  is compact.

holds.

However, if  $\int_{R_0}^{+\infty} \frac{dx_1}{g^3(x_1)} = +\infty$ , we cannot pass to a limit because the right hand side of (5.19) is growing. Therefore, we have to control the Dirichlet integral of the vector field  $\mathbf{v}^{(l)}$  over subdomains  $\Omega_k \subset \Omega_l$ , for  $k \leq l$ . To do this we apply the special techniques (so called estimates of Saint Venant type) developed by V.A. Solonnikov and O.A. Ladyzhenskaya (see [19], [33]). Let us estimate the norm  $\|\nabla \mathbf{v}^{(l)}\|_{L^2(\Omega_k)}$  with  $k < l$ . We introduce the function

$$\mathbf{U}_k^{(l)}(x) = \begin{cases} \mathbf{v}^{(l)}(x), & x \in \Omega_k, \\ \theta_k(x)\mathbf{v}^{(l)}(x) + \widehat{\mathbf{v}}_k^{(l)}(x), & x \in \omega_k, \\ 0, & x \in \Omega \setminus \Omega_{k+1}, \end{cases} \quad (5.20)$$

where  $\theta_k(x)$  is a smooth even in  $x_2$  cut-off function with the following properties:

$$\begin{aligned} \theta_k(x) &= \begin{cases} 1, & x \in \Omega_k, \\ 0, & x \in \Omega \setminus \Omega_{k+1}, \end{cases} \\ |\nabla \theta_k(x)| &\leq \frac{c}{g(R_k)}. \end{aligned} \quad (5.21)$$

Let  $\widehat{\mathbf{v}}_k^{(l)}$  be a solution of the problem

$$\begin{aligned} \operatorname{div} \widehat{\mathbf{v}}_k^{(l)} &= -\nabla \theta_k \cdot \mathbf{v}^{(l)} & \text{in } \omega_k, \\ \widehat{\mathbf{v}}_k^{(l)} &= 0 & \text{on } \partial \omega_k. \end{aligned} \quad (5.22)$$

Since

$$\int_{\omega_k} \nabla \theta_k \cdot \mathbf{v}^{(l)} dx = \int_{\omega_k} \operatorname{div} (\theta_k \mathbf{v}^{(l)}) dx = \int_{\partial \omega_k} \theta_k \mathbf{v}^{(l)} \cdot \mathbf{n} dx = \int_{\sigma(R_k)} \mathbf{v}^{(l)} \cdot \mathbf{n} dx = 0,$$

a solution  $\widehat{\mathbf{v}}_k^{(l)}$  of problem (5.22) exists and satisfies the estimate

$$\|\nabla \widehat{\mathbf{v}}_k^{(l)}\|_{L^2(\omega_k)} \leq c \|\nabla \theta_k \cdot \mathbf{v}^{(l)}\|_{L^2(\omega_k)}, \quad (5.23)$$

where  $c$  is independent of  $k$  (see Lemma 5.4). Using the estimate (5.21) and the Poincaré inequality (5.6), from (5.23) we derive the estimate

$$\|\nabla \widehat{\mathbf{v}}_k^{(l)}\|_{L^2(\omega_k)} \leq c \|\nabla \theta_k \cdot \mathbf{v}^{(l)}\|_{L^2(\omega_k)} \leq \frac{c}{g(R_k)} \|\mathbf{v}^{(l)}\|_{L^2(\omega_k)} \leq c \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}. \quad (5.24)$$

Notice that  $\widehat{\mathbf{v}}_k^{(l)}$  is not necessary symmetric, so we symmetrized it as in (4.10). For simplicity we do not change the notation of  $\widehat{\mathbf{v}}_k^{(l)}$ , i.e.  $\widehat{\mathbf{v}}_k^{(l)}$  is symmetric in the following text.

Set  $\boldsymbol{\eta} = \mathbf{U}_k^{(l)}$  in (5.12). Then, because  $\mathbf{U}_k^{(l)} = 0$  in  $\Omega \setminus \Omega_{k+1}$  and

$$\int_{\Omega_{k+1}} ((\mathbf{v}^{(l)} + \mathbf{A}) \cdot \nabla) \mathbf{U}_k^{(l)} \cdot \mathbf{U}_k^{(l)} dx = 0,$$

we obtain

$$\begin{aligned}
\nu \int_{\Omega_k} |\nabla \mathbf{v}^{(l)}|^2 dx &= \int_{\omega_k} ((\mathbf{v}^{(l)} + \mathbf{A}) \cdot \nabla) \mathbf{U}_k^{(l)} \cdot (\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)}) dx \\
&\quad - \nu \int_{\omega_k} \nabla \mathbf{v}^{(l)} : \nabla \mathbf{U}_k^{(l)} dx + \int_{\Omega_{k+1}} (\mathbf{v}^{(l)} \cdot \nabla) \mathbf{U}_k^{(l)} \cdot \mathbf{A} dx \\
&\quad - \nu \int_{\Omega_{k+1}} \nabla \mathbf{A} : \nabla \mathbf{U}_k^{(l)} dx + \int_{\Omega_{k+1}} (\mathbf{A} \cdot \nabla) \mathbf{U}_k^{(l)} \cdot \mathbf{A} dx + \int_{\Omega_{k+1}} \mathbf{f} \cdot \mathbf{U}_k^{(l)} dx.
\end{aligned} \tag{5.25}$$

In order to estimate the right hand side of (5.25), we use first the inequalities (5.24), (5.7) and the Poincaré inequality (5.6) to obtain

$$\begin{aligned}
\|\mathbf{v}^{(l)}\|_{L^4(\omega_k)} &\leq c g^{1/2}(R_k) \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}; \\
\|\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)}\|_{L^4(\omega_k)} &\leq \|\mathbf{v}^{(l)}\|_{L^4(\omega_k)} + \|\widehat{\mathbf{v}}_k^{(l)}\|_{L^4(\omega_k)} \\
&\leq c g^{1/2}(R_k) \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)} + c g^{1/2}(R_k) \|\nabla \widehat{\mathbf{v}}_k^{(l)}\|_{L^2(\omega_k)} \\
&\leq c g^{1/2}(R_k) \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}; \\
\|\nabla \mathbf{U}_k^{(l)}\|_{L^2(\omega_k)} &\leq \|\nabla(\theta_k \mathbf{v}^{(l)})\|_{L^2(\omega_k)} + \|\nabla \widehat{\mathbf{v}}_k^{(l)}\|_{L^2(\omega_k)} \\
&\leq \|\nabla \theta_k\|_{L^\infty(\omega_k)} \|\mathbf{v}^{(l)}\|_{L^2(\omega_k)} + \|\theta_k\|_{L^\infty(\omega_k)} \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)} \\
&\quad + c \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)} \leq c \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}.
\end{aligned} \tag{5.26}$$

Below we will need the following inequality

$$\int_{\omega_k} |\mathbf{A}|^2 |\mathbf{w}|^2 dx \leq c \varepsilon^2 \int_{\omega_k} |\nabla \mathbf{w}|^2 dx \quad \forall \mathbf{w} \in W_{loc}^{1,2}(\overline{\Omega}), \quad \mathbf{w} = 0 \text{ on } \partial\Omega, \tag{5.27}$$

which can be proved arguing like for proving Leray-Hopf's inequality.

By using the Hölder inequality, (5.26) and (5.27) we obtain

$$\begin{aligned}
&\left| \int_{\omega_k} ((\mathbf{v}^{(l)} + \mathbf{A}) \cdot \nabla) \mathbf{U}_k^{(l)} \cdot (\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)}) dx \right| \\
&\leq \left( \|\mathbf{v}^{(l)}\|_{L^4(\omega_k)} \|\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)}\|_{L^4(\omega_k)} + \|\mathbf{A}(\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)})\|_{L^2(\omega_k)} \right) \|\nabla \mathbf{U}_k^{(l)}\|_{L^2(\omega_k)} \\
&\leq c g(R_k) \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^3 + c \varepsilon \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)} \|\nabla(\mathbf{v}^{(l)} - \mathbf{U}_k^{(l)})\|_{L^2(\omega_k)} \\
&\leq c g(R_k) \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^3 + c \varepsilon \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^2.
\end{aligned} \tag{5.28}$$

We estimate the second term of the equation (5.25) by using the Cauchy-Schwarz inequality and the estimates (5.26):

$$\nu \left| \int_{\omega_k} \nabla \mathbf{v}^{(l)} : \nabla \mathbf{U}_k^{(l)} dx \right| \leq \nu \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)} \|\nabla \mathbf{U}_k^{(l)}\|_{L^2(\omega_k)} \leq \nu c \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^2. \tag{5.29}$$

To estimate the third term of (5.25) we use the Leray-Hopf inequality (4.20), the Hölder inequality, (5.26) and (5.27):

$$\begin{aligned}
& \left| \int_{\Omega_{k+1}} (\mathbf{v}^{(l)} \cdot \nabla) \mathbf{U}_k^{(l)} \cdot \mathbf{A} \, dx \right| \\
& \leq \left| \int_{\Omega_k} (\mathbf{v}^{(l)} \cdot \nabla) \mathbf{v}^{(l)} \cdot \mathbf{A} \, dx \right| + \left| \int_{\omega_k} (\mathbf{v}^{(l)} \cdot \nabla) \mathbf{U}_k^{(l)} \cdot \mathbf{A} \, dx \right| \\
& \leq c\varepsilon \|\nabla \mathbf{v}^{(l)}\|_{L^2(\Omega_k)}^2 + \|\nabla \mathbf{U}_k^{(l)}\|_{L^2(\omega_k)} \left( \int_{\omega_k} |\mathbf{v}^{(l)}|^2 |\mathbf{A}|^2 \, dx \right)^{1/2} \\
& \leq c\varepsilon \left( \|\nabla \mathbf{v}^{(l)}\|_{L^2(\Omega_k)}^2 + \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^2 \right).
\end{aligned} \tag{5.30}$$

The last three terms of (5.25) are estimated by using the Hölder inequality, the Cauchy inequality, (5.17), (5.18) and (5.26):

$$\begin{aligned}
& \nu \left| \int_{\Omega_{k+1}} \nabla \mathbf{A} : \nabla \mathbf{U}_k^{(l)} \, dx \right| + \left| \int_{\Omega_{k+1}} (\mathbf{A} \cdot \nabla) \mathbf{U}_k^{(l)} \cdot \mathbf{A} \, dx \right| + \left| \int_{\Omega_{k+1}} \mathbf{f} \cdot \mathbf{U}_k^{(l)} \, dx \right| \\
& \leq c \left( \|\nabla \mathbf{A}\|_{L^2(\Omega_{k+1})} + \|\mathbf{A}\|_{L^4(\Omega_{k+1})}^2 + \|\mathbf{f}\|_{H^*(\Omega_{k+1})} \right) \|\nabla \mathbf{U}_k^{(l)}\|_{L^2(\Omega_{k+1})} \\
& \leq \frac{c}{2\mu} \left( \|\nabla \mathbf{A}\|_{L^2(\Omega_{k+1})} + \|\mathbf{A}\|_{L^4(\Omega_{k+1})}^2 + \|\mathbf{f}\|_{H^*(\Omega_{k+1})} \right)^2 + \frac{c\mu}{2} \|\nabla \mathbf{U}_k^{(l)}\|_{L^2(\Omega_{k+1})}^2 \\
& \leq \frac{2c}{\mu} \left( \|\nabla \mathbf{A}\|_{L^2(\Omega_{k+1})}^2 + \|\mathbf{A}\|_{L^4(\Omega_{k+1})}^4 + \|\mathbf{f}\|_{H^*(\Omega_{k+1})}^2 \right) + \frac{c\mu}{2} \|\nabla \mathbf{v}^{(l)}\|_{L^2(\Omega_{k+1})}^2 \\
& \leq \frac{c(\mathbf{a}, \|\mathbf{f}\|_*)}{\mu} \left( 1 + \int_{R_0}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} \right) + c \frac{\mu}{2} \left( \|\nabla \mathbf{v}^{(l)}\|_{L^2(\Omega_k)}^2 + \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^2 \right).
\end{aligned} \tag{5.31}$$

Therefore, from (5.25), (5.28), (5.30), (5.31) it follows that

$$\begin{aligned}
& \nu \int_{\Omega_k} |\nabla \mathbf{v}^{(l)}|^2 \, dx \leq c g(R_k) \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^3 + c\varepsilon \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^2 + c\nu \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^2 \\
& + c \left( \varepsilon + \frac{\mu}{2} \right) \left( \|\nabla \mathbf{v}^{(l)}\|_{L^2(\Omega_k)}^2 + \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^2 \right) + \frac{c(\mathbf{a}, \|\mathbf{f}\|_*)}{\mu} \left( 1 + \int_{R_0}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} \right).
\end{aligned}$$

For  $\varepsilon$  and  $\mu$  sufficiently small, we obtain

$$\begin{aligned}
& \int_{\Omega_k} |\nabla \mathbf{v}^{(l)}|^2 \, dx \leq c g(R_k) \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^3 + c \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^2 \\
& + c(\mathbf{a}, \|\mathbf{f}\|_*) \left( 1 + \int_{R_0}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} \right).
\end{aligned} \tag{5.32}$$

Using the remark 3.1 several times we derive

$$\begin{aligned} \int_{R_k}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} &\leq \int_{R_k}^{R_{k+1}} \frac{dx_1}{\left(\frac{1}{2}g(R_k)\right)^3} = \frac{8(R_{k+1} - R_k)}{g^3(R_k)} = \frac{4}{L g^2(R_k)}, \\ \int_{R_{k-1}}^{R_k} \frac{dx_1}{g^3(x_1)} &\geq \int_{R_{k-1}}^{R_k} \frac{dx_1}{\left(\frac{3}{2}g(R_{k-1})\right)^3} = \frac{8(R_k - R_{k-1})}{27 g^3(R_{k-1})} = \frac{4}{27 L g^2(R_{k-1})}. \end{aligned}$$

Since  $g(R_k) \geq \frac{1}{2}g(R_{k-1})$  we get

$$\int_{R_{k-1}}^{R_k} \frac{dx_1}{g^3(x_1)} \geq \frac{1}{27 L g^2(R_k)}.$$

It follows that

$$\int_{R_k}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} \leq \frac{4}{L g^2(R_k)} = 27 \cdot 4 \frac{1}{27 L g^2(R_k)} \leq 27 \cdot 4 \int_{R_{k-1}}^{R_k} \frac{dx_1}{g^3(x_1)}.$$

Thus we have

$$\int_{R_0}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} \leq \int_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)} + \int_{R_k}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} \leq 109 \int_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)}$$

and the inequality (5.32) becomes

$$\int_{\Omega_k} |\nabla \mathbf{v}^{(l)}|^2 dx \leq c g(R_k) \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^3 + c \|\nabla \mathbf{v}^{(l)}\|_{L^2(\omega_k)}^2 + c(\mathbf{a}, \|\mathbf{f}\|_*) \left(1 + \int_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)}\right).$$

Denote  $y_k = \int_{\Omega_k} |\nabla \mathbf{v}^{(l)}|^2 dx$ . Since  $\int = \int_{\omega_k} - \int_{\Omega_{k+1}}$ , we can rewrite the last inequality as

$$y_k \leq c_*(y_{k+1} - y_k) + c_{**}g(R_k)(y_{k+1} - y_k)^{3/2} + \frac{1}{2}Q_k, \quad (5.33)$$

where

$$Q_k = 2 c(\mathbf{a}, \|\mathbf{f}\|_*) \left(1 + \int_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)}\right). \quad (5.34)$$

We have, using the remark 3.1 again

$$\begin{aligned} &c_*(Q_{k+1} - Q_k) + c_{**}g(R_k)(Q_{k+1} - Q_k)^{3/2} \\ &\leq 2 c_* c(\mathbf{a}, \|\mathbf{f}\|_*) \int_{R_k}^{R_{k+1}} \frac{dx_1}{g^3(x_1)} + c_{**}g(R_k) \left(c(\mathbf{a}, \|\mathbf{f}\|_*) \int_{R_k}^{R_{k+1}} \frac{dx_1}{g^3(x_1)}\right)^{3/2} \\ &\leq c_1 c(\mathbf{a}, \|\mathbf{f}\|_*) g^{-2}(R_k) \leq c_2 c(\mathbf{a}, \|\mathbf{f}\|_*) \int_{R_{k-1}}^{R_k} \frac{dx_1}{g^3(x_1)} \leq c(\mathbf{a}, \|\mathbf{f}\|_*) \left(1 + \int_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)}\right) = \frac{1}{2}Q_k \end{aligned}$$

for  $k$  large enough if  $\frac{\int_{R_{k-1}}^{R_k} \frac{dx_1}{g^3(x_1)}}{1 + \int_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)}} \rightarrow 0$  when  $k \rightarrow +\infty$ .

**Claim.** Let non negative numbers  $y_k$ ,  $k = 1, \dots, N$ , satisfy the inequalities  $y_{k+1} \geq y_k$  and

$$y_k \leq c_*(y_{k+1} - y_k) + c_{**}g(R_k)(y_{k+1} - y_k)^{3/2} + \frac{1}{2}Q_k, \quad (5.35)$$

where  $c_*, c_{**}, Q_k$  are non negative numbers such that

$$\frac{1}{2}Q_k \geq c_*(Q_{k+1} - Q_k) + c_{**}g(R_k)(Q_{k+1} - Q_k)^{3/2}. \quad (5.36)$$

If  $N < +\infty$  and  $y_N \leq Q_N$  then  $y_k \leq Q_k \quad \forall k < N$ .

Although this claim is proved in [33], for the reader convenience we give the proof which is based on induction. Suppose we have proved that  $y_{k+1} \leq Q_{k+1}$ . If  $y_k > Q_k$  then  $0 \leq y_{k+1} - y_k < Q_{k+1} - Q_k$ . Since the function  $\tau \rightarrow F(\tau) = c_*\tau + c_{**}g(R_k)\tau^{3/2}$  is increasing we get

$$y_k \leq F(y_{k+1} - y_k) + \frac{1}{2}Q_k < F(Q_{k+1} - Q_k) + \frac{1}{2}Q_k \leq \frac{1}{2}Q_k + \frac{1}{2}Q_k = Q_k$$

and a contradiction. Thus,  $y_k \leq Q_k$ .

Since  $Q_k$  satisfies the condition (5.36), the inequality (5.33) together with (5.19) and the claim above, the estimate

$$y_k = \int_{\Omega_k} |\nabla \mathbf{v}^{(l)}|^2 dx \leq c(\mathbf{a}, \|\mathbf{f}\|_*) \left( 1 + \int_{R_0}^{R_k} \frac{dx_1}{g^3(x_1)} \right) \quad \forall k \leq l \quad (5.37)$$

holds.

Since for every bounded domain  $\Omega_k$ ,  $k > 0$  the embedding  $W_S^{1,2}(\Omega_k) \hookrightarrow L_S^4(\Omega_k)$  is compact, the estimate (5.37) is sufficient to assure the existence of a subsequence  $\{\mathbf{v}^{(l_m)}\}$  which converges weakly in  $\dot{W}_S^{1,2}(\Omega_k)$  and strongly in  $L_S^4(\Omega_k)$  for any  $k > 0$ . Such subsequence could be constructed by Cantor diagonal process: we can choose a weakly convergent subsequence  $\{\mathbf{v}^{(l_m)}\}$  in  $\dot{W}_S^{1,2}(\Omega_1)$  which converges strongly in  $L_S^4(\Omega_1)$ . In the same manner we subtract a subsequence of  $\{\mathbf{v}^{(l_m)}\}$  in  $\Omega_2$  which we call also  $\{\mathbf{v}^{(l_m)}\}$  for the sake of simplicity. Continuing this we can choose a desired subsequence. Taking in integral identity (5.12) an arbitrary test function  $\boldsymbol{\eta}$  with a compact support, we can find a number  $k$  such that  $\text{supp } \boldsymbol{\eta} \subset \Omega_k$  and  $\boldsymbol{\eta} \in H_S(\Omega_k)$ . Extending  $\boldsymbol{\eta}$  by zero into  $\Omega \setminus \Omega_k$ , and considering all integrals in (5.12) as integrals over  $\Omega$ , we can pass in (5.12) to a limit as  $l_m \rightarrow +\infty$ . As a result we get for the limit vector function  $\mathbf{v}$  the integral identity (5.2). Therefore,  $\mathbf{u} = \mathbf{A} + \mathbf{v}$  is a weak solution of problem (1.1). The estimate (5.5) for  $\mathbf{v}$  follows from (5.37). Since for  $\mathbf{A}$  the analogous to (5.5) is also valid, we obtain (5.5) for the sum  $\mathbf{u} = \mathbf{A} + \mathbf{v}$ .

**Remark 5.3.** If the norms  $\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}$  and  $\|\mathbf{f}\|_*$  are sufficiently small, it can be proved using the methods proposed in [19] and [33] that the weak solution  $\mathbf{u}$  is unique in a class of functions with the Dirichlet integral growing “not too fast”.

**Remark 5.4.** If  $D$  is a channel-like outlet and  $|\mathbb{F}|$  is sufficiently small, it can be proved using the methods from [19] and [33] that the weak solution  $\mathbf{u}$  tends to the Poiseuille flow as  $x_1 \rightarrow +\infty$ . In this sense our result extends the result obtained by H. Morimoto and H. Fujita in [21], [22].

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